Fluctuations in a Thermal Field and Dissipation of a Black Hole Spacetime: Far-Field Limit

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We study the backreaction of a thermal field in a weak gravitational background depicting the far-field limit of a black hole enclosed in a box by the closed time path (CTP) effective action and the influence functional method. We derive the noise and dissipation kernels of this system in terms of quantities in quasiequilibrium, and formally prove the existence of a fluctuation-dissipation relation (FDR) at all temperatures between the quantum fluctuations of the thermal radiance and the dissipation of the gravitational field. This dynamical self-consistent interplay between the quantum field and the classical spacetime is, we believe, the correct way to treat backreaction problems. To emphasize this point we derive an Einstein–Langevin equation which describes the nonequilibrium dynamics of the gravitational perturbations under the influence of the thermal field. We show the connection between our method and the linear response theory (LRT), and indicate how the functional method can provide more accurate results than prior derivations of FDRs via LRT in the test-field, static conditions. This method is in principle useful for treating fully nonequilibrium cases such as backreaction in black hole collapse.

1. INTRODUCTION

In a recent essay [1] one of us outlined the program of black hole fluctuations and backreaction we are pursuing using stochastic semiclassical gravity [2] theory based on the Schwinger–Keldysh effective action [3] and the Feynman–Vernon influence functional [4] methods. We mentioned prior works for static, quasistatic, and dynamic black hole spacetimes and commented on how to improve on their shortcomings. In this paper, following the cues suggested in that essay, we discuss first the static case of a Schwarzschild black hole, with a further simplification of taking the far-field limit, and

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derive the fluctuation-dissipation relation (FDR) [5] under these conditions. In our view (following Sciama's [6] hints) the FDR embodies the backreaction of Hawking radiance [7]. In a recent work [8] we treated a relativistic thermal plasma in a weak gravitational field. Since the far-field limit of a Schwarzschild metric is just the perturbed Minkowski spacetime, the results there are useful for our present problem. The more complicated case of near-horizon limit is also doable by calculating the fluctuations of the energy-momentum tensor for the quantum field (with the help of, e.g., the Page approximation [9]). Derivation of the dissipation and noise kernels for a static black hole in a cavity is currently under investigation [10, 11].

1.1. Fluctuation and Backreaction in Static Black Holes

We recapitulate what was said before [1] on the state of the art for problems in this case. Backreaction in this context usually refers to seeking a consistent solution of the semiclassical Einstein equation for the geometry of a black hole in equilibrium with its Hawking radiation (enclosed in a box to ensure relative stability). Much effort in the last 15 years has been devoted to finding a regularized energy-momentum for the backreaction calculation. (See refs. 12 and 13 for recent status and earlier references.) Some important early work on backreaction was carried out by Bardeen and by Hajicek and Israel [14] and York [15], and more recently by Massar and by Parentani and Piran [16] along similar lines.

Since the quantum field in such problems is assumed to be in a Hartle–Hawking state, concepts and techniques from thermal field theory are useful. Hartle and Hawking [17] and Gibbons and Perry [18] used the periodicity condition of the Green function on the Euclidean section to give a simple derivation of the Hawking temperature for a Schwarzschild black hole. The most relevant work to our present problem is that by Mottola [19], who showed that in some generalized Hartle–Hawking states a FDR exists between the expectation values of the commutator and anticommutator of the energy-momentum tensor. This FDR has a form familiar in linear response theory [20]:

$$N_{abcd}(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \coth\left(\frac{1}{2}\beta\omega\right) \tilde{D}_{abcd}(\mathbf{x}, \mathbf{x}'; \omega) \tag{1}$$

where N and D are the anticommutator and commutator functions of the energy-momentum tensor, respectively ($\tilde{D}$ is the temporal Fourier transform of D). That is,

$$N_{abcd}(x, x') = \langle\{\hat{T}_{ab}(x), \hat{T}_{cd}(x')\}\rangle_\beta$$

$$D_{abcd}(x, x') = \langle[\hat{T}_{ab}(x), \hat{T}_{cd}(x')]\rangle_\beta \tag{2}$$

He also identifies the two-point function D as a dissipation kernel by relating it to the time rate of change of the energy density when the metric is slightly perturbed. Thus, Eq. (1) represents a bona fide FDR relating the fluctuations of a certain quantity (say, energy density) to the time rate of change of the very same quantity.

However, this type of FDR has rather restricted significance as it is based on the assumption of a specific background spacetime (static in this case) and state (thermal) of the matter field(s). It is not adequate for the description of backreaction where the spacetime and the state of matter are determined in a self-consistent manner by their dynamics and mutual influence. We should therefore look for a FDR for a parametric family of metrics (belonging to a general class) and a more general state of the quantum matter (in particular, for Boulware and Unruh states). We expect the derivation of such a FDR will be more complicated than the simple case above where the Green functions are periodic in imaginary time throughout (not just as an initial condition), and where one can simply take the results of linear response theory in thermal equilibrium (for all times) almost verbatim.

Even in this simple case, it is worthwhile to note that there is a small departure from standard linear response theory for quantum systems. This arises from the observation that the dissipation kernel in usual linear response analyses is given by a two-point commutator function of the underlying quantum field, which is independent of the quantum state for free field theory. In this case, we are still restricted to free fields in a curved background. However, since the dissipation now depends on a two-point function of the stress tensor, it is a four-point function of the field, with appropriate derivatives and coincidence limits. This function is, in general, state dependent. We have seen examples from related cosmological backreaction problems [2] where it is possible to explicitly relate the dissipation to particle creation in the field, which is definitely a state-dependent process. For the black-hole case, this would imply a quantum-state-dependent damping of semiclassical perturbations. To obtain a causal FDR for states more general than the Hartle–Hawking state, one needs to use the in-in (or Schwinger–Keldysh) formalism applied to a class of quasistatic metrics (generalization of York [15]) and calculate the fluctuations of the energy-momentum tensor for the noise kernel. In our problem such a calculation with backreaction is carried out in full detail, albeit only for a weak gravitational field here which depicts the far-field limit of a Schwarzschild black hole spacetimes. We wish to address the thermal field aspects of the problem, while saving the geometric aspects in the near-horizon case for a later investigation.

1.2. Thermal Fields in Linear Gravity

The behavior of a relativistic quantum field at finite temperature in a weak gravitational field has been studied by Gross *et al.* [21], Rebhan and coworkers [22], and Brandt and coworkers [23] for scalar and Abelian gauge fields. In these works, the thermal graviton polarization tensor and the effective action have been calculated and applied to the study of the stability of hot flat/curved spaces and "dynamics" of cosmological perturbations. To describe screening effects and the stability of thermal quantum gravity, one needs only the real part of the polarization tensor, but for damping effects, the imaginary part is essential. The gravitational polarization tensor obtained from the thermal graviton self-energy represents only a part (the thermal correction to the vacuum polarization) of the finite-temperature quantum stress tensor. There are in general also contributions from particle creation (from vacuum fluctuations at zero and finite temperatures). These processes engender dissipation in the dynamics of the gravitational field and their fluctuations appear as noise in the thermal field. We aim at finding a relation, the FDR, between these two processes, which embodies the backreaction self-consistently.

In this work we use open system concepts and functional methods à la Schwinger–Keldysh [3] and Feynman–Vernon [4]. By casting the effective action in the form of an influence functional we derive the noise and dissipation kernels explicitly and prove that they satisfy a fluctuation-dissipation relation (FDR) [5] at all temperatures. We also derive a stochastic semiclassical equation for the nonequilibrium dynamics of the gravitational field under the influence of the thermal radiance.

We adopt the Hartle–Hawking picture where the black hole is bathed eternally—actually in quasithermal equilibrium—in the Hawking radiance it emits. It will be described here by a massless scalar quantum field at the Hawking temperature. As is well known this quasiequilibrium condition is possible only if the black hole is enclosed in a box of size slightly larger than the event horizon [15] (or embedded in an anti-de Sitter space [24]). In the asymptotic limit, the gravitational field is described by a linear perturbation from Minkowski spacetime. In equilibrium the thermal bath can be characterized by a relativistic fluid with a four-velocity (timelike normalized vector field) u^μ and temperature in its own rest frame β^{-1}. Taking into account the four-velocity u^μ of the fluid, a manifestly Lorentz-covariant approach to thermal field theory may be used [25]. However, in order to simplify the involved tensorial structure we work in the comoving coordinate system of the fluid where $u^\mu = (1, 0, 0, 0)$.

By making conformal transformations on the field and the spacetime, our results may be easily generalized to the case of a conformally coupled

quantum scalar field at finite temperature in a spatially flat Friedmann–Robertson–Walker universe [26]. Indeed we earlier used the functional method and the Brownian motion paradigm [4] to study similar problems in semiclassical gravity [27]. We found that quantum noise arising from fluctuations in the particle creation would constitute a stochastic source (whose effect can overdominate the expectation value of the energy-momentum tensor in the semiclassical Einstein equation [28]) in a new form of Einstein–Langevin equation [2]. We also came to the understanding that backreaction of vacuum quantum field processes (such as particle creation) on the dynamics of the early universe near the Planck time is summarily a manifestation of a FDR in semiclassical gravity.

In Section 2 we describe our model and the derivation of the thermal CTP effective action. We compare it with the influence action [4] and identify the dissipation and the noise kernels representing the linear response of the gravitational field and the quantum fluctuations of the thermal radiance, respectively. In Section 3 we show that they obey a fluctuation-dissipation relation at all temperatures. In Section 4 we show how to derive a stochastic semiclassical equation for the gravitational perturbations from this effective action, which depicts the nonequilibrium dynamics of the gravitational field in a thermal radiation bath.

2. CTP EFFECTIVE ACTION AT FINITE TEMPERATURE

2.1. The Model

In this section, we derive the CTP effective action for a thermal quantum field in a classical gravitational background. To describe the radiation we consider a free massless scalar field ϕ arbitrarily coupled to a gravitational field $g_{\mu\nu}$ with classical action

$$S_m[\phi, g_{\mu\nu}] = -\frac{1}{2} \int d^n x \sqrt{-g} \, [g^{\mu\nu} \, \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2] \tag{3}$$

where $R(x)$ is the scalar curvature and the arbitrary parameter $\xi(n)$ defines the type of coupling between the scalar field and the gravitational field. If $\xi = (n - 2)/[4(n - 1)]$, where n is the spacetime dimensions, the field is said to be conformally coupled; if $\xi = 0$, the quantum field is said to be minimally coupled. In the weak-field limit we consider a small perturbation $h_{\mu\nu}$ from flat spacetime $\eta_{\mu\nu}$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{4}$$

with signature $(-, +, \cdots, +)$ for the Minkowski metric. Using this metric

and neglecting the surface terms that appear in an integration by parts, we can write the action for the scalar field as

$$S_m[\phi, h_{\mu\nu}] = \frac{1}{2} \int d^n x \ \phi[\Box + V^{(1)} + V^{(2)} + \cdots]\phi \tag{5}$$

where the first- and second-order perturbative operators $V^{(1)}$ and $V^{(2)}$ are given by

$$V^{(1)} \equiv -\{[\partial_\mu \bar{h}^{\mu\nu}(x)]\partial_\nu + \bar{h}^{\mu\nu}(x) \ \partial_\mu \ \partial_\nu + \xi R^{(1)}(x)\}$$

$$V^{(2)} \equiv \left\{ [\partial_\mu \hat{h}^{\mu\nu}(x)]\partial_\nu + \hat{h}^{\mu\nu}(x) \ \partial_\mu \ \partial_\nu - \xi \left(R^{(2)}(x) + \frac{1}{2} h(x) R^{(1)}(x) \right) \right\} \tag{6}$$

In the above expressions, $R^{(k)}$ is the k-order term in the pertubation $h_{\mu\nu}(x)$ of the scalar curvature and the definitions $\bar{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ denote a linear and a quadratic combination of the perturbation, respectively,

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h\eta_{\mu\nu}$$

$$\hat{h}_{\mu\nu} \equiv h_\mu^\alpha h_{\alpha\nu} - \frac{1}{2} h h_{\mu\nu} + \frac{1}{8} h^2 \eta_{\mu\nu} - \frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} \eta_{\mu\nu} \tag{7}$$

For the gravitational field we take the following action:

$$S_g^{\text{div}}[g_{\mu\nu}] = \frac{1}{l_P^{n-2}} \int d^n x \ \sqrt{-g} R(x)$$

$$+ \frac{\alpha \bar{\mu}^{n-4}}{4(n-4)} \int d^n x \ \sqrt{-g}$$

$$\times \left[3R_{\mu\nu\alpha\beta}(x) \ R^{\mu\nu\alpha\beta}(x) - \left(1 - 360 \left(\xi - \frac{1}{6} \right)^2 \right) R(x)R(x) \right] \tag{8}$$

The first term is the classical Einstein–Hilbert action and the second divergent term in four dimensions is the counterterm used in order to renormalize the effective action. In this action $l_P^2 = 16\pi G$, $\alpha = (2880\pi^2)^{-1}$, and $\bar{\mu}$ is an arbitrary mass scale. It is noteworthy that the counterterms are independent of the temperature because the thermal contribution to the effective action is finite and does not include additional divergences.

2.2. CTP Effective Action

The CTP effective action at finite temperature for a free quantum scalar field in a gravitational background is given by

$$\Gamma^{\beta}_{\text{CTP}}[h^{\pm}_{\mu\nu}] = S^{\text{div}}_g[h^+_{\mu\nu}] - S^{\text{div}}_g[h^-_{\mu\nu}] - \frac{i}{2}\,\text{Tr}\{\ln\,\bar{G}^{\beta}_{ab}[h^{\pm}_{\mu\nu}]\} \qquad (9)$$

where $a, b = \pm$ denote the forward and backward time paths and $\bar{G}^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ is the complete 2×2 matrix propagator with thermal boundary conditions for the differential operator $\Box + V^{(1)} + V^{(2)} + \cdots$. Although the actual form of $\bar{G}^{\beta}_{ab}$ cannot be explicitly given, it is easy to obtain a perturbative expansion in terms of $V^{(k)}_{ab}$, the k-order matrix version of the complete differential operator defined by $V^{(k)}_{\pm\pm} \equiv \pm V^{(k)}_{\pm}$ and $V^{(k)}_{\pm\mp} \equiv 0$, and G^{β}_{ab}, the thermal matrix propagator for a massless scalar field in flat spacetime [8]. To second order $\bar{G}^{\beta}_{ab}$ reads

$$\bar{G}^{\beta}_{ab} = G^{\beta}_{ab} - G^{\beta}_{ac}V^{(1)}_{cd}G^{\beta}_{db} - G^{\beta}_{ac}V^{(2)}_{cd}G^{\beta}_{db} + G^{\beta}_{ac}V^{(1)}_{cd}G^{\beta}_{de}V^{(1)}_{ef}G^{\beta}_{fb} + \cdots \qquad (10)$$

Expanding the logarithm and dropping one term independent of the perturbation $h^{\pm}_{\mu\nu}(x)$, we can write the CTP effective action perturbatively as

$$\Gamma^{\beta}_{\text{CTP}}[h^{\pm}_{\mu\nu}]$$
$$= S^{\text{div}}_g[h^+_{\mu\nu}] - S^{\text{div}}_g[h^-_{\mu\nu}]$$
$$\quad + \frac{i}{2}\,\text{Tr}[V^{(1)}_+ G^{\beta}_{++} - V^{(1)}_- G^{\beta}_{--} + V^{(2)}_+ G^{\beta}_{++} - V^{(2)}_- G^{\beta}_{--}]$$
$$\quad - \frac{i}{4}\,\text{Tr}[V^{(1)}_+ G^{\beta}_{++}V^{(1)}_+ G^{\beta}_{++} + V^{(1)}_- G^{\beta}_{--}V^{(1)}_- G^{\beta}_{--}$$
$$\quad - 2V^{(1)}_+ G^{\beta}_{+-}V^{(1)}_- G^{\beta}_{-+}] \qquad (11)$$

In computing the traces, some terms containing divergences are canceled using counterterms introduced in the classical gravitational action after dimensional regularization. In general, the nonlocal pieces are of the form $\text{Tr}[V^{(1)}_a G^{\beta}_{mn}V^{(1)}_b G^{\beta}_{rs}]$. In terms of the Fourier transformed thermal propagators $\tilde{G}^{\beta}_{ab}(k)$ these traces can be written as

$$\text{Tr}[V^{(1)}_a G^{\beta}_{mn}V^{(1)}_b G^{\beta}_{rs}]$$
$$= \int d^n x\, d^n x'\; h^a_{\mu\nu}(x)h^b_{\alpha\beta}(x')$$
$$\quad \times \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\, e^{ik\cdot(x-x')}\tilde{G}^{\beta}_{mn}(k+q)\tilde{G}^{\beta}_{rs}(q)\text{T}^{\mu\nu,\alpha\beta}(q,k) \qquad (12)$$

where the tensor $\text{T}^{\mu\nu,\alpha\beta}(q, k)$ is defined in ref. 8 after an expansion in terms of a basis of 14 tensors [22]. In particular, the last trace of (11) may be split into two different kernels $\text{N}^{\mu\nu,\alpha\beta}(x - x')$ and $\text{D}^{\mu\nu,\alpha\beta}(x - x')$,

$$\frac{i}{2} \operatorname{Tr}[V_+^{(1)} G_{+-}^\beta - V_-^{(1)} G_{-+}^\beta]$$

$$= -\int d^4x \, d^4x' \, h_{\mu\nu}^+(x) h_{\alpha\beta}^-(x')[D^{\mu\nu,\alpha\beta}(x - x') + iN^{\mu\nu,\alpha\beta}(x - x')] \qquad (13)$$

One can express the Fourier transforms of these kernels, respectively, as

$$\tilde{N}^{\mu\nu,\alpha\beta}(k) = \pi^2 \int \frac{d^4q}{(2\pi)^4} \{\theta(k^o + q^o)\theta(-q^o)$$

$$+ \theta(-k^o - q^o)\theta(q^o) + n_\beta(|q^o|) + n_\beta(|k^o + q^o|)$$

$$+ 2n_\beta(|q^o|)n_\beta(|k^o + q^o|)\}\delta(q^2)\delta[(k + q)^2]T^{\mu\nu,\alpha\beta}(q, k) \qquad (14)$$

$$D^{\mu\nu,\alpha\beta}(k) = -i\pi^2 \int \frac{d^4q}{(2\pi)^4} \{\theta(k^o + q^o)\theta(-q^o)$$

$$- \theta(-k^o - q^o)\theta(q^o) + sg(k^o + q^o)n_\beta(|q^o|)$$

$$- sg(q^o)n_\beta(|k^o + q^o|)\}\delta(q^2)\delta[(k + q)^2]T^{\mu\nu,\alpha\beta}(q, k) \qquad (15)$$

Using the property $T^{\mu\nu,\alpha\beta}(q, k) = T^{\mu\nu,\alpha\beta}(-q, -k)$, it is easy to see that $N^{\mu\nu,\alpha\beta}(x - x')$ is symmetric and $D^{\mu\nu,\alpha\beta}(x - x')$ antisymmetric in their arguments; that is, $N^{\mu\nu,\alpha\beta}(x) = N^{\mu\nu,\alpha\beta}(-x)$ and $D^{\mu\nu,\alpha\beta}(x) = -D^{\mu\nu,\alpha\beta}(-x)$.

To identify properly the physical meanings of these kernels we have to write the renormalized CTP effective action at finite temperature (11) in an influence functional form [4]. Here N, the imaginary part of the CTP effective action, can be identified with the noise kernel and D, the antisymmetric piece of the real part, with the dissipation kernel. In Section 3 we will see that these kernels thus identified indeed satisfy a thermal FDR.

If we denote the difference and the sum of the perturbations $h_{\mu\nu}^\pm$ defined along each branch $C_\pm$ of the complex time path of integration C by $[h_{\mu\nu}] \equiv h_{\mu\nu}^+ - h_{\mu\nu}^-$ and $\{h_{\mu\nu}\} \equiv h_{\mu\nu}^+ + h_{\mu\nu}^-$, respectively, we can write the influence functional form of the thermal CTP effective action to second order in $h_{\mu\nu}$ as

$$\Gamma_{\text{CTP}}^\beta[h_{\mu\nu}^\pm] \simeq \frac{1}{2l_P^2} \int d^4x \, d^4x' [h_{\mu\nu}](x) \, L_{(o)}^{\mu\nu,\alpha\beta}(x - x')\{h_{\alpha\beta}\}(x')$$

$$+ \frac{1}{2} \int d^4x \, [h_{\mu\nu}](x) \, T_{(\beta)}^{\mu\nu}$$

$$+ \frac{1}{2} \int d^4x \, d^4x' \, [h_{\mu\nu}](x) \, H^{\mu\nu,\alpha\beta}(x - x')\{h_{\alpha\beta}\}(x')$$

$$- \frac{1}{2} \int d^4x \, d^4x' \, [h_{\mu\nu}](x) \, D^{\mu\nu,\alpha\beta}(x - x')\{h_{\alpha\beta}\}(x')$$

$$+ \frac{i}{2} \int d^4x \, d^4x' \, [h_{\mu\nu}](x) \, N^{\mu\nu,\alpha\beta}(x - x')[h_{\alpha\beta}](x') \qquad (16)$$

The first line is the Einstein–Hilbert action to second order in the perturbation $h_{\mu\nu}^{\pm}(x)$. Here $L_{(o)}^{\mu\nu,\alpha\beta}(x)$ is a symmetric kernel [i.e., $L_{(o)}^{\mu\nu,\alpha\beta}(x) = L_{(o)}^{\mu\nu,\alpha\beta}(-x)$] and its Fourier transform is given by

$$L_{(o)}^{\mu\nu,\alpha\beta}(k) = \frac{1}{4} [-k^2 T_1^{\mu\nu,\alpha\beta}(q, k) + 2k^2 T_4^{\mu\nu,\alpha\beta}(q, k)$$

$$+ T_8^{\mu\nu,\alpha\beta}(q, k) - 2T_{13}^{\mu\nu,\alpha\beta}(q, k)] \qquad (17)$$

The 14 elements of the tensor basis $T_i^{\mu\nu,\alpha\beta}(q, k)$ ($i = 1, \ldots, 14$) are defined in ref. 22. In the second line $T^{\mu\nu}{}_{(\beta)}$ has the form of a perfect fluid stress-energy tensor

$$T_{(\beta)}^{\mu\nu} = \frac{\pi^2}{30\beta^2} \left[u^\mu u^\nu + \frac{1}{3} (\eta^{\mu\nu} + u^\mu u^\nu) \right] \qquad (18)$$

where u^μ is the four-velocity of the plasma and the factor $\pi^2/30\beta^4$ is the familiar thermal energy density for massless scalar particles at temperature β^{-1}. In the third line, the Fourier transform of the symmetric kernel $H^{\mu\nu,\alpha\beta}(x)$ can be expressed as

$$H^{\mu\nu,\alpha\beta}(k) = -\frac{\alpha k^4}{4} \left\{ \frac{1}{2} \ln \frac{|k^2|}{\mu^2} Q^{\mu\nu,\alpha\beta}(k) + \frac{1}{3} \bar{Q}^{\mu\nu,\alpha\beta}(k) \right\}$$

$$+ \frac{\pi^2}{180\beta^4} \{-T_1^{\mu\nu,\alpha\beta}(u, k) - 2T_2^{\mu\nu,\alpha\beta}(u, k)$$

$$+ T_4^{\mu\nu,\alpha\beta}(u, k) + 2T_5^{\mu\nu,\alpha\beta}(u, k)\}$$

$$+ \frac{\xi}{96\beta^2} \{k^2 T_1^{\mu\nu,\alpha\beta}(u, k) - 2k^2 T_4^{\mu\nu,\alpha\beta}(u, k)$$

$$- T_8^{\mu\nu,\alpha\beta}(u, k) + 2T_{13}^{\mu\nu,\alpha\beta}(u, k)\}$$

$$+ \pi \int \frac{d^4q}{(2\pi)^4} \left\{ \delta(q^2) n_\beta(|q^o|) \mathcal{P} \left[\frac{1}{(k + q)^2} \right] \right.$$

$$+ \left. \delta[(k + q)^2] n_\beta(|k^o + q^o|) \mathcal{P} \left[\frac{1}{q^2} \right] \right\} T^{\mu\nu,\alpha\beta}(q, k) \qquad (19)$$

where μ is a simple redefinition of the renormalization parameter $\bar{\mu}$ given by $\mu \equiv \bar{\mu} \exp(23/15 + \frac{1}{2} \ln 4\pi - \frac{1}{2}\gamma)$, and the tensors $Q^{\mu\nu,\alpha\beta}(k)$ and $\bar{Q}^{\mu\nu,\alpha\beta}(k)$ are defined, respectively, by

$$Q^{\mu\nu,\alpha\beta}(k) = \frac{3}{2} \left\{ T_1^{\mu\nu,\alpha\beta}(q,k) - \frac{1}{k^2} T_8^{\mu\nu,\alpha\beta}(q,k) + \frac{2}{k^4} T_{12}^{\mu\nu,\alpha\beta}(q,k) \right\}$$

$$- \left[1 - 360 \left(\xi - \frac{1}{6} \right)^2 \right]$$

$$\times \left\{ T_4^{\mu\nu,\alpha\beta}(q,k) + \frac{1}{k^4} T_{12}^{\mu\nu,\alpha\beta}(q,k) - \frac{1}{k^2} T_{13}^{\mu\nu,\alpha\beta}(q,k) \right\} \qquad (20)$$

$$\bar{Q}^{\mu\nu,\alpha\beta}(k) = \left[1 + 576 \left(\xi - \frac{1}{6} \right)^2 - 60 \left(\xi - \frac{1}{6} \right)(\lambda - 36\xi') \right]$$

$$\times \left\{ T_4^{\mu\nu,\alpha\beta}(q,k) + \frac{1}{k^4} T_{12}^{\mu\nu,\alpha\beta}(q,k) - \frac{1}{k^2} T_{13}^{\mu\nu,\alpha\beta}(q,k) \right\} \qquad (21)$$

In the above equation ξ' means the derivative of $\xi(n)$ with respect to n evaluated at $n = 4$. $\bar{H}^{\mu\nu,\alpha\beta}(k)$ is the complete contribution of a free massless quantum scalar field to the thermal graviton polarization tensor [22, 23] and it is responsible for the instabilities found in flat spacetime at finite temperature [21–23]. Equation (19) reflects the fact that the kernel $\bar{H}^{\mu\nu,\alpha\beta}(k)$ has thermal as well as nonthermal contributions. Note that it reduces to the first term in the zero-temperature limit ($\beta \to \infty$)

$$\bar{H}^{\mu\nu,\alpha\beta}(k) \simeq -\frac{\alpha k^4}{4} \left\{ \frac{1}{2} \ln \frac{|k^2|}{\mu^2} Q^{\mu\nu,\alpha\beta}(k) + \frac{1}{3} \bar{Q}^{\mu\nu,\alpha\beta}(k) \right\} \qquad (22)$$

and at high temperatures the leading term (β^{-4}) may be written as

$$\bar{H}^{\mu\nu,\alpha\beta}(k) \simeq \frac{\pi^2}{30\beta^4} \sum_{i=1}^{14} H_i(r) T_i^{\mu\nu,\alpha\beta}(u,K) \qquad (23)$$

where we have introduced the dimensionless external momentum $K^\mu \equiv k^\mu/|\vec{k}| \equiv (r, \hat{k})$. The $H_i(r)$ coefficients were first given in ref. 22 and generalized to the next to leading order (β^{-2}) in ref. 23. (They are given with the MTW sign convention in ref. 8.)[3]

Finally, as defined above, $N^{\mu\nu,\alpha\beta}(x)$ is the noise kernel representing the random fluctuations of the thermal radiance and $D^{\mu\nu,\alpha\beta}(x)$ is the dissipation kernel, describing the dissipation of energy of the gravitational field.

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[3] Note that to leading order, the addition of the contribution of other kinds of matter fields to the effective action, even graviton contributions, does not change the tensor structure of this kernel and only the overall factors are different [22].

3. FLUCTUATION-DISSIPATION RELATION AND LINEAR RESPONSE THEORY

3.1. Fluctuation-Dissipation Relation

These two kernels found above are functionally related by a fluctuation-dissipation relation (FDR). This relation reflects the balance between the quantum fluctuations in the thermal radiance and the energy loss by the gravitational field. In ref. 8 we have shown explicitly how this relation appears at zero temperature and at high temperature. Here, using the properties of the thermal propagators, we show that the FDR is formally satisfied for all temperatures.

We begin by showing that the FDR naively appears at the level of propagators as a direct consequence of the KMS relation [29]. Then, using a generalization of this KMS relation, we see how the FDR is also satisfied by our noise and dissipation kernels.

To obtain the FDR at the level of propagators we need to introduce the Schwinger and the Hadamard propagators. These propagators are defined as the thermal average of the anticommutator $G(x - x') \equiv -i\langle[\phi(x), \phi(x')]\rangle_\beta$ and the commutator $G_\beta^{(1)}(x - x') \equiv -i\langle\{\phi(x), \phi(x')\}\rangle_\beta$, respectively. The first represents the linear response of a relativistic system to an external perturbation and the second the random fluctuations of the system itself [29, 5]. Since we can write the KMS condition satisfied by the propagators G_{+-}^β and G_{-+}^β in Fourier space as

$$\tilde{G}_{-+}^\beta(k) = e^{\beta k^o} \tilde{G}_{+-}^\beta(k) \tag{24}$$

the Fourier transform of both the Schwinger $\tilde{G}(k)$ and the Hadamard $\tilde{G}_\beta^{(1)}(k)$ propagators can be expressed, for example, in terms of $\tilde{G}_{+-}^\beta(k)$ alone

$$\tilde{G}(k) \equiv \tilde{G}_{-+}^\beta(k) - \tilde{G}_{+-}^\beta(k) = (e^{\beta k^o} - 1)\, \tilde{G}_{+-}^\beta(k)$$

$$\tilde{G}_\beta^{(1)}(k) \equiv \tilde{G}_{-+}^\beta(k) + \tilde{G}_{+-}^\beta(k) = (e^{\beta k^o} + 1)\, \tilde{G}_{+-}^\beta(k) \tag{25}$$

The FDR satisfied by these propagators follows inmediately from the above equalities [29, 5]:

$$\tilde{G}_\beta^{(1)}(k) = \coth\left(\frac{\beta k^o}{2}\right)\tilde{G}(k) \tag{26}$$

Obviously, this relation can also be recovered if we write the explicit expressions for the Fourier transform of the propagators

$$\tilde{G}(k) = -2\pi i\, sg(k^o)\delta(k^2)$$

$$\tilde{G}_\beta^{(1)}(k) = -2\pi i\, \coth\left(\frac{\beta|k^o|}{2}\right)\delta(k^2) \tag{27}$$

To use this last approach in our case could be a very difficult task because one needs to compute the integrals for the noise and dissipation kernels explicitly. On the other hand, if we follow the first technique, we only need to generalize the KMS condition of Eq. (24) to the product of two propagators. This generalization reads

$$\tilde{G}^{\beta}_{-+}(k + q)\tilde{G}^{\beta}_{+-}(q) = e^{\beta k^o} \tilde{G}^{\beta}_{+-}(k + q)\tilde{G}^{\beta}_{-+}(q) \tag{28}$$

and can be used to deduce the following formal identity:

$$\tilde{G}^{\beta}_{-+}(k + q)\tilde{G}^{\beta}_{+-}(q) + \tilde{G}^{\beta}_{+-}(k + q)\tilde{G}^{\beta}_{-+}(q)$$
$$= \coth\left(\frac{\beta k^o}{2}\right)[\tilde{G}^{\beta}_{-+}(k + q)\,\tilde{G}^{\beta}_{+-}(q) - \tilde{G}^{\beta}_{+-}(k + q)\tilde{G}^{\beta}_{-+}(q)] \tag{29}$$

Finally, one only needs to write, from the trace of Eq. (12) and the definitions (14) and (15), the noise and dissipation kernels in terms of the propagators $\tilde{G}^{\beta}_{\pm\mp}$, respectively, as

$$\bar{N}^{\mu\nu,\alpha\beta}(k) = -\frac{1}{4} \int \frac{d^4q}{(2\pi)^4} [\tilde{G}^{\beta}_{-+}(k + q)\tilde{G}^{\beta}_{+-}(q)$$
$$+ \tilde{G}^{\beta}_{+-}(k + q)\,\tilde{G}^{\beta}_{-+}(q)]T^{\mu\nu,\alpha\beta}(q, k) \tag{30}$$

$$D^{\mu\nu,\alpha\beta}(k) = \frac{i}{4} \int \frac{d^4q}{(2\pi)^4} [\tilde{G}^{\beta}_{-+}(k + q)\tilde{G}^{\beta}_{+-}(q)$$
$$- \tilde{G}^{\beta}_{+-}(k + q)\tilde{G}^{\beta}_{-+}(q)]\,T^{\mu\nu,\alpha\beta}(q, k) \tag{31}$$

and use the formal equality (29) to prove that they are related by the thermal identity

$$\bar{N}^{\mu\nu,\alpha\beta}(k) = i \coth\left(\frac{\beta k^o}{2}\right) D^{\mu\nu,\alpha\beta}(k) \tag{32}$$

In coordinate space we have the analogous expression

$$N^{\mu\nu,\alpha\beta}(x) = \int d^4x'\, K_{FD}(x - x')D^{\mu\nu,\alpha\beta}(x') \tag{33}$$

where the fluctuation-dissipation kernel $K_{FD}(x - x')$ is given by the integral

$$K_{FD}(x - x') = i \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \coth\left(\frac{\beta k^o}{2}\right) \tag{34}$$

The proof of this FDR at finite temperature is in some sense formal because we have assumed along the argument that the integrals are always well

defined. Nevertheless, the exact results obtained for the zero- and high-temperature limits [8] indicate that the noise and dissipation kernels are well-defined distributions [30]. The asymptotic analysis has also been useful to determine the physical origin of the fluctuations.

3.2. Linear Response Theory

A fluctuation-dissipation relation is usually derived using linear response theory (LRT). We now show the connection between the LRT [20] and the functional methods we have used here. In the spirit of LRT the gravitational field is considered as a weak external source which imparts disturbances to the radiance whose response is studied to linear order.

Let us first recall the main features of LRT. Consider a system described by the Hamiltonian operator $\hat{H}_o$ initially coupled linearly to an external driving agent, say A_α. Since we are only interested in how the system responds to the external agent, and not the details of the agent, we will ignore the Hamiltonian for the external perturbation, but write the complete operator Hamiltonian of the sytem as

$$\hat{H} = \hat{H}_o + A_\alpha \hat{J}^\alpha \qquad (35)$$

where J^α is the current operator associated with the external agent. If the system is in thermal equilibrium before the external source is applied, the first-order response of the system to this external force is given by the thermal expectation value of the commutator of the current operator over its thermal average

$$\langle [J^\mu(x) - \langle J^\mu(x)\rangle_\beta, J^\nu(x') - \langle J^\nu(x')\rangle_\beta]\rangle_\beta \qquad (36)$$

In contrast, the intrinsic quantum fluctuations of the system are described by the thermal average of the anticommutator. In our case, the conserved current operator is given by the stress-energy tensor $T^{\mu\nu}(x)$ as derived from the classical action. Our objective is to show that the response and fluctuation functions for the stress-energy tensor considered in the LRT are equivalent to the dissipation and noise kernels, respectively.

First, we write the classical action for the matter field to linear order in the gravitational perturbations,

$$S_m[\phi, h_{\mu\nu}] \simeq \frac{1}{2} \int d^4x \, [\phi\Box\phi + h_{\mu\nu}T^{\mu\nu}] \qquad (37)$$

with the stress-energy tensor given by

$$T^{\mu\nu} = P^{\mu\nu,\alpha\beta} \, \partial_\alpha\phi \, \partial_\beta\phi + \xi(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\phi^2 \qquad (38)$$

Note that $T^{\mu\nu}(x)$ is conserved if the classical unperturbed equation of motion

for ϕ is satisfied and it reduces to the stress-energy tensor for a scalar field in flat spacetime if $\xi = 0$. Alternatively, we can write the Hamiltonian formulation of our problem. If we introduce the conjugate momentum variable of the matter field to first order in the perturbation

$$\Pi \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \sim \dot{\phi} + \frac{1}{2} h_{\mu\nu} \frac{\partial T^{\mu\nu}}{\partial \dot{\phi}} \tag{39}$$

we can write the Hamiltonian as

$$H \simeq \frac{1}{2} \int d^3\vec{x} \left[\Pi^2 + (\vec{\nabla}\phi)^2 - h_{\mu\nu} T^{\mu\nu} \right] \tag{40}$$

Note that to first order $\dot{\phi}$ and Π are interchangeable in the expression for the stress-energy tensor.

Using the thermal version of the Wick theorem [31, 32], one can write, after some algebra, the equilibrium thermal average of the two-point function for the stress-energy tensor operator at different spacetime points in terms of products of thermal propagators,

$$\langle T^{\mu\nu}(x), T^{\alpha\beta}(x') \rangle_\beta - \langle T^{\mu\nu}(x) \rangle_\beta \langle T^{\alpha\beta}(x') \rangle_\beta$$

$$= -2 \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \int \frac{d^4q}{(2\pi)^4} \tilde{G}^\beta_{-+}(k+q) \tilde{G}^\beta_{+-}(q) \mathrm{T}^{\mu\nu,\alpha\beta}(q,k) \tag{41}$$

Finally, defining $\Delta_\beta T^{\mu\nu}(x) \equiv T^{\mu\nu}(x) - \langle T^{\mu\nu}(x) \rangle_\beta$ and using the expressions for the noise and dissipation kernels given in (30) and (31), respectively, we obtain

$$\langle \{ \Delta_\beta T^{\mu\nu}(x), \Delta_\beta T^{\alpha\beta}(x') \} \rangle_\beta = 8 \mathrm{N}^{\mu\nu,\alpha\beta}(x-x') \tag{42}$$

$$\langle [\Delta_\beta T^{\mu\nu}(x), \Delta_\beta T^{\alpha\beta}(x')] \rangle_\beta = 8i \mathrm{D}^{\mu\nu,\alpha\beta}(x-x') \tag{43}$$

From these formal identities we conclude that the functional method gives a description of the lowest order dynamics of a near-equilibrium system equivalent to that given traditionally by the LRT.

4. EINSTEIN–LANGEVIN EQUATION

To reinforce the points made at the beginning about the dynamic nature of the backreaction of thermal radiance on the black hole spacetime even for the quasistatic case, we now derive from the thermal CTP effective action a dynamical equation governing the dissipative evolution of the gravitational field under the influence of the fluctuations of the thermal radiance. It is in the form of an Einstein–Langevin equation [2].

We first introduce the influence functional [4] $\mathscr{F} \equiv \exp(iS_{\text{IF}})$, where the influence action S_{IF} is related to the CTP effective action in the semiclassical limit by [2]

$$\mathscr{F} = \exp i\left(\text{Re}\{\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}]\} \right.$$
$$\left. + \frac{i}{2} \int d^4x \, d^4x' \, [h_{\mu\nu}](x)N^{\mu\nu,\alpha\beta}(x-x')[h_{\alpha\beta}](x') \right) \quad (44)$$

where $\text{Re}\{\cdot\}$ denotes taking the real part. Following ref. 4, we can interpret the real part of the influence functional as the characteristic functional of a nondynamical stochastic variable $j^{\mu\nu}(x)$,

$$\Phi([h_{\mu\nu}]) = \exp\left(-\frac{1}{2} \int d^4x \, d^4x' \, [h_{\mu\nu}](x)N^{\mu\nu,\alpha\beta}(x-x')[h_{\alpha\beta}](x') \right) \quad (45)$$

This classical stochastic field represents probabilistically the quantum fluctuations of the matter field and is responsible for the dissipation of the gravitational field. By definition, the above characteristic functional is the functional Fourier transform of the probability distribution functional $\mathscr{P}[j^{\mu\nu}]$ with respect to $j^{\mu\nu}$,

$$\Phi([h_{\mu\nu}]) = \int \mathscr{D}j^{\mu\nu} \, \mathscr{P}[j^{\mu\nu}] \exp\left[i \int d^4x \, [h_{\mu\nu}](x)j^{\mu\nu}(x) \right] \quad (46)$$

Using (45), one can easily see that the probability distribution functional is related to the noise kernel by the formal expression

$$\mathscr{P}[j^{\mu\nu}]$$
$$= \frac{\exp(-\frac{1}{2} \int d^4x \, d^4x' \, j_{\mu\nu}(x)[N^{\mu\nu,\alpha\beta}(x-x')]^{-1}j_{\alpha\beta}(x'))}{\int \mathscr{D}j^{\mu\nu} \exp(-\frac{1}{2} \int d^4x \, d^4x' \, j_{\mu\nu}(x)[N^{\mu\nu,\alpha\beta}(x-x')]^{-1}j_{\alpha\beta}(x'))} \quad (47)$$

For an arbitrary functional of the stochastic field $\mathscr{E}[j^{\mu\nu}]$, the average value with respect to the previous probability distribution functional is defined as the functional integral $\langle\mathscr{E}[j^{\mu\nu}]\rangle_j \equiv \int \mathscr{D}[j^{\mu\nu}] \, \mathscr{P}[j^{\mu\nu}]\mathscr{E}[j^{\mu\nu}]$. In terms of this stochastic average the influence functional can be written as $\mathscr{F} = \langle\exp(i\Gamma^{\text{st}}_{\text{CTP}}[h^{\pm}_{\mu\nu}])\rangle_j$, where $\Gamma^{\text{st}}_{\text{CTP}}[h^{\pm}_{\mu\nu}]$ is the modified effective action

$$\Gamma^{\text{st}}_{\text{CTP}}[h^{\pm}_{\mu\nu}] \equiv \text{Re}\{\Gamma^{\beta}_{\text{CTP}}[h^{\pm}_{\mu\nu}]\} + \int d^4x \, [h_{\mu\nu}](x)j^{\mu\nu}(x) \quad (48)$$

Clearly, because of the quadratic nature of the characteristic functional (45) and its relation to the probability distribution functional (46), the field $j^{\mu\nu}(x)$

is a zero-mean Gaussian stochastic variable. This means that its two-point correlation function, which is given in terms of the noise kernel by

$$\langle j^{\mu\nu}(x)j^{\alpha\beta}(x')\rangle_j = N^{\mu\nu,\alpha\beta}(x-x') \tag{49}$$

completely characterizes the stochastic process. The Einstein–Langevin equation follows from taking the functional derivative of the stochastic effective action (48) with respect to $[h_{\mu\nu}](x)$ and imposing $[h_{\mu\nu}](x) = 0$. In our case, this leads to

$$\frac{1}{l_P^2}\int d^4x'\, L_{(o)}^{\mu\nu,\alpha\beta}(x-x')h_{\alpha\beta}(x')$$

$$+ \frac{1}{2}T_{(\beta)}^{\mu\nu} + \int d^4x'\,(H^{\mu\nu,\alpha\beta}(x-x') - D^{\mu\nu,\alpha\beta}(x-x'))h_{\alpha\beta}(x') + j^{\mu\nu}(x) = 0 \tag{50}$$

To obtain a simpler and clearer expression we can rewrite this stochastic equation for the gravitational perturbation in the harmonic gauge $\bar{h}_{,\nu}^{\mu\nu} = 0$,

$$\Box\bar{h}^{\mu\nu}(x) + l_P^2 \Bigg\{ T_{(\beta)}^{\mu\nu} + 2P_{\rho\sigma,\alpha\beta}\int d^4x'\,(H^{\mu\nu,\alpha\beta}(x-x')$$

$$- D^{\mu\nu,\alpha\beta}(x-x'))\,\bar{h}^{\rho\sigma}(x') + 2j^{\mu\nu}(x) \Bigg\} = 0 \tag{51}$$

where we have used the definition for $\bar{h}^{\mu\nu}(x)$ written in (7) and the tensor $P_{\rho\sigma,\alpha\beta}$ is given by

$$P_{\rho\sigma,\alpha\beta} = \frac{1}{2}(\eta_{\rho\alpha}\eta_{\sigma\beta} + \eta_{\rho\beta}\eta_{\sigma\alpha} - \eta_{\rho\sigma}\eta_{\alpha\beta}) \tag{52}$$

Note that this differential stochastic equation includes a nonlocal term responsible for the dissipation of the gravitational field and a noise source term which accounts for the fluctuations in the thermal radiance. They are connected by a FDR as described in the last section. Note also that this equation in combination with the correlation for the stochastic variable (49) determine the two-point correlation for the stochastic metric fluctuations $\langle\bar{h}_{\mu\nu}(x)\bar{h}_{\alpha\beta}(x')\rangle_j$ self-consistently.

5. CONCLUSIONS

In this paper we show how the functional methods can be used effectively to study the nonequilibrium dynamics of a weak classical gravitational field in a thermal quantum field. The closed time path (CTP) effective action and

the influence functional were used to derive the noise and dissipation kernels of this system. The backreaction of the thermal radiance on the gravitational field is embodied in a fluctuation-dissipation relation (FDR), which connects the fluctuations in the thermal radiation and the energy dissipation of the gravitational field. We prove formally the existence of such a relation for thermal fields at all temperatures.

We also show the formal equivalence of this method with linear response theory (LRT) for lowest order perturbation of a near-equilibrium system, and how the response functions such as the contribution of the quantum scalar field to the thermal graviton polarization tensor can be derived. An important quantity not usually obtained in LRT but of equal importance manifest in the CTP approach is the noise term arising from the quantum and statistical fluctuations in the thermal field.

Finally, we emphasize that the backreaction is intrinsically a dynamic process which traditional LRT calculations cannot capture fully. We illustrate this point by deriving an Einstein–Langevin equation for the nonequilibrium dynamics of the gravitational field with backreaction from the thermal field. This method can be applied to quasidynamic [6, 33] or fully dynamic problems such as black hole collapse [14, 16, 34]. To complete the present problem of quasistatic black hole backreaction, we need to perform the same calculation for the full Schwarzschild spacetime. Currently we are working on the fluctuations of the energy-momentum tensor near the black hole horizon and the derivation of the noise kernel. Results will be reported in future publications.

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